

Absence of a second order phase transition in $\lambda\phi^4$ theory

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Abstract

We calculate the self-energy at finite temperature in scalar $\lambda\phi^4$ theory to second order in a modified perturbation expansion. Using the renormalisation group equation to tame the logarithms in momentum, it gives an equation to determine the critical temperature. Due to the infrared freedom of the theory, this equation is satisfied, irrespective of the value of the temperature. We conclude that there is no second order phase transition in this theory.

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1 Introduction

In this work we examine the possibility of a second order phase transition in the scalar field theory with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4, \quad (1)$$

where $m^2 < 0$. We have to calculate the effective mass at finite temperature in the symmetric phase. As the temperature is lowered, this mass may vanish at a definite temperature, indicating a second order phase transition [1-3].

A general problem here is the breakdown of usual perturbation expansion when powers of temperature compensate for powers of coupling constant. In the scalar theory it is due to the generation of the thermal mass, which is generally taken into account by a summation over the so-called ring or daisy diagrams [2,4]. An alternative and more consistent method is to modify the perturbation expansion by adding a (temperature dependent) mass term in the free Lagrangian and a compensating counterterm in the interaction [3,5-7]. Thus we rewrite (1) as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}M^2\phi^2 - \frac{\lambda}{4!}\phi^4 - \frac{1}{2}(m^2 - M^2)\phi^2 \quad (2)$$

The perturbation series will be a joint expansion in powers of λ and $(m^2 - M^2)$. Although $(m^2 - M^2)$ may not be small, it will appear in a combination which would be small. The parameter M becomes the effective mass, if we require perturbative corrections to the mass term (at zero momentum) to be zero. In general all these methods of taking the thermal mass into account are also equally effective in removing the infrared divergences of a massless field theory at finite temperature.

To be specific, let us denote the self-energy at momentum p_μ and temperature T , calculated upto a certain order of our perturbation expansion, by $\Sigma(p^2, M, T)$. (We suppress its dependence on m , λ and the renormalisation scale μ to be introduced below.) Then the effective mass M is obtained by solving

$$\Sigma(p^2 = 0, M, T) = 0. \quad (3)$$

Formally the critical temperature T_c of a second order phase transition is attained when M goes to zero,

$$\Sigma(p^2 = 0, M = 0, T_c) = 0. \quad (4)$$

The other problem, specific to our analysis, is the new infrared divergence encountered in (4): If we first set $p_\mu = 0$ and calculate Σ perturbatively beyond the first order, there arises powers of $\ln M^2$ (multiplied with T^2 or m^2) [5]. Thus $\Sigma(p^2 = 0, M, T)$ diverges logarithmically as $M \rightarrow 0$ and (4) is not meaningful. In the same way, if we first set $M = 0$, then $\Sigma(p^2, M = 0, T)$ contain powers of $\ln p^2$ and again we cannot reach the limit in (4).

As the pure scalar theory is infrared free, it is possible to sum these infrared divergences in perturbation expansion by the renormalisation group equation. The equation we use is identical to the one at zero temperature.³ But due to the presence of an additional scale, viz,

³We are not using the scheme where the parameters in the Lagrangian are renormalised not only at a momentum scale μ but also at a temperature T_0 , say, as in Ref.[8].

the temperature, the solution generates new logarithmic singularities. They are, however, tamed by multiplication with the running coupling constant. In sec.4 we shall set $M = 0$ and sum the leading logarithms in p^2 .

2 Self energy to second order

To make the cancellation of ultraviolet divergences explicit in the calculation, we augment the Lagrangian density (1) with the renormalisation counterterms

$$\mathcal{L}_{ct} = \frac{1}{2}A(\partial_\mu\phi)^2 - \frac{1}{2}m^2B\phi^2 - \frac{\lambda}{4!}\mu^{2\epsilon}C\phi^4 \quad (5)$$

where A, B, and C are the divergent coefficients. In the minimal subtraction scheme of dimensional regularisation in dimension d , they are, to order λ^2 [9],

$$A = -\frac{\hat{\lambda}^2}{24\epsilon}, \quad B = \frac{\hat{\lambda}}{2\epsilon} + \hat{\lambda}^2\left(\frac{1}{2\epsilon^2} - \frac{1}{4\epsilon}\right), \quad C = \frac{3\hat{\lambda}}{2\epsilon} + \frac{3}{4}\hat{\lambda}\left(\frac{3}{\epsilon^2} - \frac{2}{\epsilon}\right), \quad (6)$$

where $\hat{\lambda} = \lambda/16\pi^2$, $\epsilon = 2 - d/2$ and μ is the renormalisation scale. Corresponding to the modified scheme (2), the total Lagrangian splits as

$$\mathcal{L} + \mathcal{L}_{ct} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}M^2\phi^2 - \frac{\lambda}{4!}\mu^{2\epsilon}\phi^4 + \frac{1}{2}A(\partial_\mu\phi)^2 - \frac{1}{2}M^2B\phi^2 - \frac{\lambda}{4!}\mu^{2\epsilon}C\phi^4 - \frac{1}{2}(m^2 - M^2)(1+B)\phi^2 \quad (7)$$

Because of our use of the minimal subtraction scheme, where the renormalisation constants are mass independent, the Lagrangian (7) is, except for the last term, the same as the sum of Lagrangians (1) and (5) with m^2 replaced by M^2 . The last term incorporates its own renormalisation counterterm.

We use the real-time formulation of the finite temperature field theory [10], where the thermal propagator becomes a 2×2 matrix. Working below threshold, we need only calculate the 11-component of the Σ -matrix, which we have already denoted by Σ .

The self-energy diagrams to second order are shown in Fig.1. Digrams (a) and (c) are specific to our modified perturbation expansion. It will be observed that, except for the diagram (e), all others are independent of momentum and so particularly simple to evaluate [2-7,9,11]. To begin with, we do not restrict either the external momentum p_μ or the effective mass M to zero. Denoting these contributions by $-i\Sigma$, the diagrams (a) and (b) contribute respectively

$$\Sigma_{(a)} = (m^2 - M^2)(1 + B), \quad \Sigma_{(b)} = G(M^2), \quad (8)$$

where

$$G(M^2) = \frac{\lambda}{2} \left(i \int \frac{(dl)}{k^2 - M^2 + i\epsilon} + 2\pi \int (dl) n(\omega_l) \delta(l^2 - M^2) \right) \quad (9)$$

(dl) standing for $d^d l / (2\pi)^d$. Isolating the divergent piece, we write

$$G(M^2) = -\frac{\hat{\lambda}M^2}{2\epsilon} + \bar{G}(M^2). \quad (10)$$

\bar{G} has the high temperature expansion (M/T small) [2],

$$\bar{G}(M^2) = \frac{\lambda}{2} \left\{ \frac{T^2}{12} - \frac{MT}{4\pi} + \frac{M^2}{8\pi^2} \left(\ln \frac{T}{\mu} + \text{const.} \right) + O\left(\frac{M^4}{T^2}\right) \right\} \quad (11)$$

The two integrals in (9) separately contain $M^2 \ln M^2$ terms, which cancel out in the sum \bar{G} . Using the mass derivative formula [12], diagrams (c) and (d) can be evaluated as

$$\Sigma_{(c)} = (m^2 - M^2)(1 + B) \frac{\partial}{\partial M^2} G, \quad \Sigma_{(d)} = G \frac{\partial}{\partial M^2} G \quad (12)$$

We next evaluate the diagram (e) of Fig.1,

$$\begin{aligned} \Sigma_{(e)} = & -\frac{\lambda^2}{6} \int \int (dk)(dl) \frac{1}{k^2 - M^2 + i\epsilon} \frac{1}{l^2 - M^2 + i\epsilon} \frac{1}{(p - k - l)^2 - M^2 + i\epsilon} \\ & + i \frac{\lambda^2}{2} \int (dk) 2\pi \delta(k^2 - M^2) n(\omega_k) \int (dl) \frac{1}{l^2 - M^2 + i\epsilon} \frac{1}{(p - k - l)^2 - M^2 + i\epsilon} \\ & + \frac{\lambda^2}{2} \int \int (dk)(dl) (2\pi)^2 \delta(k^2 - M^2) \delta(l^2 - M^2) n(\omega_k) n(\omega_l) \frac{1}{(p - k - l)^2 - M^2 + i\epsilon} \end{aligned} \quad (13)$$

The first term corresponds to the zero temperature contribution. Separating the divergent pieces, it becomes ($\hat{M}^2 = M^2/4\pi\mu^2$),

$$\hat{\lambda}^2 \left\{ \frac{M^2}{4\epsilon^2} + \frac{M^2}{2\epsilon} \left(\frac{3}{2} - \gamma - \ln \hat{M}^2 \right) - \frac{p^2}{24\epsilon} \right\} + F_1(p, M, \mu). \quad (14)$$

Similarly the second term may be written as

$$-\frac{\hat{\lambda}}{\epsilon} \left\{ \bar{G} - \frac{\hat{\lambda} M^2}{2} (\ln \hat{M}^2 + \gamma - 1) \right\} + F_2(p, M, \mu, T) \quad (15)$$

The third term is free from divergence, to be denoted by $F_3(p, M, T)$. The finite pieces F_1 , F_2 and F_3 are complicated functions of p^2 and M^2 and will be evaluated in sec.3 below in the required limit.

Finally the diagrams (f) are due to the renormalisation counterterms,

$$\Sigma_{(f)} = M^2 \left\{ \frac{\hat{\lambda}}{2\epsilon} + \hat{\lambda}^2 \left(\frac{1}{2\epsilon^2} - \frac{1}{4\epsilon} \right) \right\} + \frac{\hat{\lambda}^2}{24\epsilon} p^2 + \frac{\hat{\lambda}^2}{2\epsilon} M^2 \frac{\partial}{\partial M^2} G + \frac{3\hat{\lambda}}{2\epsilon} G. \quad (16)$$

One may now check that all the divergent pieces, belonging to the different diagrams, cancel out. The complete self-energy to second order in our modified perturbation expansion becomes

$$\Sigma(p^2, M, T) = (m^2 - M^2) + \bar{G} + (m^2 - M^2 + \bar{G}) \frac{\partial}{\partial M^2} \bar{G} + F \quad (17)$$

where

$$F(M) = F_1(p, M, \mu) + F_2(p, M, \mu, T) + F_3(p, M, T) \quad (18)$$

Noting the second term in (11) for \bar{G} , we see that the third term in (17) for Σ has a linear divergence as $M \rightarrow 0$.

3 Infrared divergence

As already mentioned in the Introduction, the familiar infrared divergence of a massless boson theory at finite temperature is cured by incorporating the thermal mass into the mass term of the free propagator. While such an effective mass serves as an infrared cut off in general, it itself tends to zero as one approaches the critical temperature, giving rise to the infrared problem in (17). Being analogous to that of an originally massless theory, this infrared singularity may be removed by summing over the daisy diagrams (Fig.2) of our modified perturbation expansion.⁴ Again using the mass derivative formula, this sum is easily seen to be a Taylor series [4],

$$\Sigma_{(daisy)} = \sum_{n=0}^{\infty} \frac{1}{n!} (m^2 - M^2 + \bar{G}(M^2))^n \left(\frac{\partial}{\partial M^2} \right)^n \bar{G}(M^2) = \bar{G}(m^2 + \bar{G}(M^2)) \quad (19)$$

which is to replace the second and third terms in (17).

Having gotten rid of the divergence, we may set $M = 0$ in (17) to get

$$\Sigma(p^2, M = 0, T) = \mathcal{M}^2 - \frac{\lambda T}{8\pi} \mathcal{M} + \hat{\lambda} \mathcal{M}^2 \left(\ln \frac{T}{\mu} + const \right) + F(M = 0), \quad (20)$$

where $\mathcal{M}^2 = m^2 + \lambda T^2/24$. The logarithmic term above multiplying m^2 and T^2 may be checked to be identical to those in (17). The F_i 's are of course, finite at $M = 0$, as long as $p^2 \neq 0$. Their logarithmic pieces can now be obtained without difficulty. The zero temperature contribution gives

$$F_1(p^2, M = 0, \mu) = \frac{\hat{\lambda}^2}{6} p^2 \left(\ln(\sqrt{-p^2}/\mu) + const \right) \quad (21)$$

It remains to evaluate the other two finite pieces given by

$$F_2(p^2, M = 0, T, \mu) = \lambda \hat{\lambda} \pi \int (dk) \delta(k^2) n(\omega_k) \ln \left(-\frac{(p-k)^2}{\mu^2} \right), \quad (22)$$

$$F_3(p^2, M = 0, T) = \frac{\lambda^2}{2} \int (dk) \int (dl) (2\pi)^2 n(\omega_k) n(\omega_l) \delta(k^2) \delta(l^2) \frac{1}{(p-k-l)^2 + i\epsilon} \quad (23)$$

To evaluate these for space-like p^2 , we may simplify the calculation by taking $p = (p_0, \vec{0})$, p_0 imaginary, and re-expressing the result finally in terms of $-p^2 > 0$. In this way we get,

$$F_2 = \frac{\lambda \hat{\lambda}}{24} T^2 \{ \ln(\sqrt{-p^2}/\mu) + \ln T/\mu + const \} \quad (24)$$

$$F_3 = \frac{\lambda \hat{\lambda}}{16} T^2 \{ \ln(\sqrt{-p^2}/T) + const \} \quad (25)$$

⁴It should be noted that there is no double counting here. If we had taken the thermal mass into account by summing over the daisy diagrams instead of modifying the perturbation expansion, the daisy diagrams of Fig.2 would correspond to a kind of superdaisy diagrams in that scheme.

Having calculated $F(M=0)$, we finally get,

$$\begin{aligned}\Sigma(p^2, M=0, T) &= \mathcal{M}^2 - \lambda T \mathcal{M} / 8\pi + \frac{\hat{\lambda}\lambda}{8} T^2 (\ln(\sqrt{-p^2}/\mu) - \frac{1}{6} \ln(\sqrt{-p^2}/T)) \\ &+ \hat{\lambda} m^2 (\ln(\sqrt{-p^2}/\mu) - \ln(\sqrt{-p^2}/T)) + \frac{\hat{\lambda}^2}{6} p^2 \ln(\sqrt{-p^2}/\mu) + O(\lambda^2 T^2)\end{aligned}\quad (26)$$

4 Renormalisation group method

Because of the presence of the $\ln(\sqrt{-p^2}/\mu)$ in the expression (26) for Σ , we cannot approach zero momentum. We have to improve the one-particle irreducible, two point function,

$$\Gamma^{(2)}(p, \lambda, m, \mu, T) = -i(p^2 - \Sigma(p^2, M=0, T)), \quad (27)$$

by summing the leading logarithms of the perturbation series. This may be conveniently done by using the solution of the renormalisation group equation,

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} + m \gamma_m(\lambda) \frac{\partial}{\partial m} - \gamma_\phi(\lambda) \right) \Gamma^{(2)}(p, \lambda, m, \mu, T) = 0 \quad (28)$$

The coefficients in this equation are calculated from the renormalization counterterms. To lowest order, they are

$$\beta = 3\lambda\hat{\lambda}, \quad \gamma_m = \frac{\hat{\lambda}}{2}, \quad \gamma_\phi = \frac{1}{6}\hat{\lambda}^2. \quad (29)$$

Let us first check that the μ dependence of $\Gamma^{(2)}$ as calculated above indeed satisfies the renormalization group equation [13]. We write $\Gamma^{(2)}$ as series in powers of λ , ($z = (\sqrt{-p^2}/\mu)$),

$$\begin{aligned}i\Gamma^{(2)} &= T^2 \sum_{n=1} a_n(z) \lambda^n + m^2 (-1 + \sum_{n=1} b_n(z) \lambda^n) + p^2 (1 + \sum_{n=2} d_n(z) \lambda^n) \\ &+ \text{terms independent of } z.\end{aligned}\quad (30)$$

Substituting this in (28) and equating like powers of λ for each of these series to zero, one gets simple differential equations in μ or z with the solutions,

$$\begin{aligned}a_1(z) &= c_1, \quad a_2(z) = \frac{3c_1}{16\pi^2} \ln z + c_2, \\ b_1(z) &= -\frac{1}{16\pi^2} \ln z + c_3, \quad d_2(z) = -\frac{1}{6(16\pi^2)^2} \ln z + c_4,\end{aligned}\quad (31)$$

where c_i 's are constants. From (26) we see that they are indeed satisfied.

The renormalization group equation used here is the same as the one of conventional (zero temperature) field theory. Being determined by the short distance properties of the theory, the coefficients β , γ_m and γ_ϕ do not know of any temperature, at least in the minimal subtraction scheme. It is only through the set of constants c_i in (31) that temperature enters $\Gamma^{(2)}$.

The solution of the renormalization group equation relates $\Gamma^{(2)}$ at two different scales. Let us consider $\Gamma^{(2)}(sP, \lambda, m, \mu, T)$ and take $s \rightarrow 0$ at the end ($P \neq 0$). Then we write the solution as

$$\Gamma^{(2)}(sP, \lambda, m, \mu, T) = s^2 \Gamma^{(2)}(P, \lambda, \frac{m}{s}, \frac{\mu}{s}, \frac{T}{s}) = s^2 Z(s) \Gamma^{(2)}(P, \bar{\lambda}, \frac{\bar{m}}{s}, \mu, \frac{T}{s}), \quad (32)$$

where we use dimensional analysis in the intermediate step. While λ and m are renormalised at the scale μ , $\bar{\lambda}$ and \bar{m} are referred to the scale $\bar{\mu} = s\mu$, with the relations

$$\bar{\lambda} = \frac{\lambda}{1 - 3\hat{\lambda} \ln s}, \quad \bar{m} = m \left(\frac{\bar{\lambda}}{\lambda} \right)^{\frac{1}{6}}. \quad (33)$$

$Z(s)$ given by

$$Z(s) = \exp\left[-\int_{\lambda}^{\bar{\lambda}} \frac{\gamma(\lambda')}{\beta(\lambda')} d\lambda'\right] = \exp[-(\bar{\lambda} - \lambda)/288\pi^2], \quad (34)$$

tends to a constant as $s \rightarrow 0$. Inserting our evaluation (26,27) of $\Gamma^{(2)}$ in the right hand side of (32) and choosing P_{μ} and μ such that $\sqrt{-P^2} = \mu = T$ to get rid of the (finite) logarithms, we get

$$\begin{aligned} iZ(s)^{-1} \Gamma^{(2)}(sP, \lambda, m, \mu, T) &= s^2 P^2 - \bar{m}^2 - \frac{\bar{\lambda} T^2}{24} + \frac{\bar{\lambda} T}{8\pi} \left(\bar{m}^2 + \frac{\bar{\lambda} T^2}{24} \right)^{\frac{1}{2}} \\ &+ \frac{\bar{\lambda} \ln s}{16\pi^2} \left(\bar{m}^2 + \frac{\bar{\lambda} T^2}{48} \right) + O(\lambda^2 T^2). \end{aligned} \quad (35)$$

We see that although terms with $\ln s$ arising from $\ln(\sqrt{-p^2}/\mu)$ are absorbed in the parameters $\bar{\lambda}$ and \bar{m} , new $\ln s$ terms arise from $\ln(\sqrt{-p^2}/T)$ present in Σ . Though they cannot lead to a logarithmic divergence as $s \rightarrow 0$, because of multiplication with $\bar{\lambda}$ (and \bar{m}), we see that there will arise terms proportional to $\bar{\lambda}$ and \bar{m} from each order of the perturbation expansion. They will form a series in powers of $\bar{\lambda} \ln s / 16\pi^2 \rightarrow -1/3$ as $s \rightarrow 0$.

The point to observe is that each term in the expression (35) for $\Gamma^{(2)}$ goes to zero as the momentum tends to zero. Retaining only the first two leading terms for small s , we have

$$\begin{aligned} \Sigma(p^2 = s^2 P^2, M = 0, T) &= -i \Gamma^{(2)}(sP, \lambda, m, \mu, T) \\ &= Z[\bar{m}^2(1 + \frac{1}{3} + \dots) + \frac{\bar{\lambda} T^2}{24}(1 + \frac{1}{6} + \dots) + O(\ln s)^{-\frac{7}{6}}], \end{aligned} \quad (36)$$

where dots denote contributions beyond the second order.

5 Conclusion

We see that equation (4), which is supposed to give a definite value for the critical temperature, is trivially satisfied, independently of the value of the temperature. We conclude that there is no proper second order phase transition in the pure scalar field theory. It is not difficult to see that the same conclusion would have been reached if we had first put

$p_\mu = 0$ and then summed the resulting series in $\ln M^2$ by the renormalisation group equation. Clearly our result is a direct consequence of the infrared freedom of the scalar field theory in dimension $d = 4$.

In dimensions $d < 4$ ($\epsilon > 0$), of interest in condensed matter physics, the scalar field theory develops a stable, non-zero infrared fixed point, at least in the perturbative calculation [14]. There one gets a finite critical temperature for a second order phase transition. But in condensed matter physics, the scalar field theory is used phenomenologically and predicts behaviour of correlation functions at or near the critical point but not the value of the critical temperature itself.

Going beyond the scalar field theory, we may say that there cannot also be a second order phase transition in any other system whose contents are described by an infrared free theory, e.g. the combined system of the Higgs and QED.

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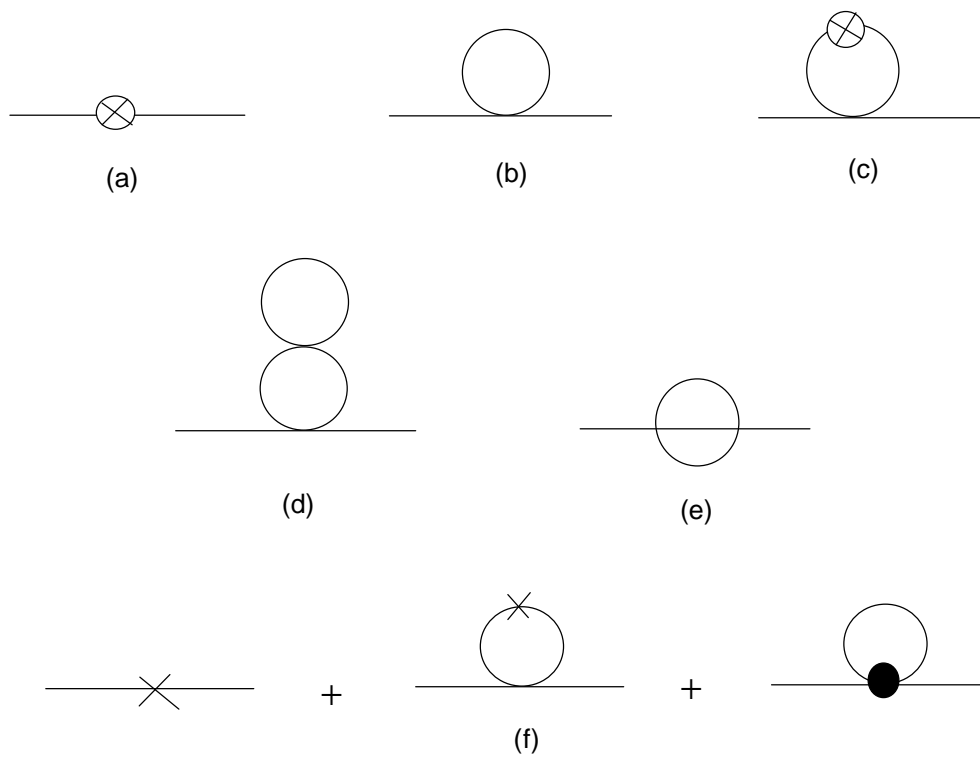


Fig.1. Self energy diagrams upto second order . Cross and solid dot denote vertices for renormalisation counterterms of ϕ^2 and ϕ^4 type respectively, while circled cross represents the additional vertex in our modified perturbation expansion.

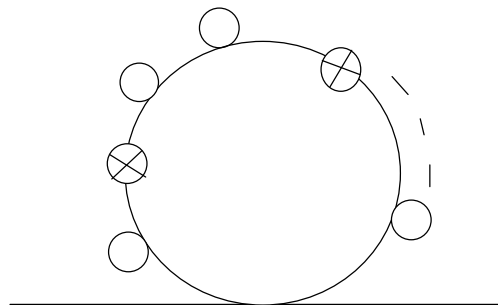


Fig.2. The so-called daisy diagram in n-th order. The counterterm insertions are not shown.